Linear Algebra I

19/12/2013, Thursday, 14:00-16:00

You are **NOT** allowed to use any type of calculators.

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1 (6+6+6+6+6=30 pts) **Linear equations**

Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 5 \\ 2 & 2 & 0 & -2 & 10 \\ -1 & 0 & -1 & 2 & -5 \\ 1 & 8 & -7 & 6 & -1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ -6 \end{pmatrix}$$

where α is a real number. Consider the linear equation $Ax = b$.

(a) Determine the *lead* and *free* variables.

(b) Determine all values of α for which the equation has *infinitely many solutions*.

(c) Determine all values of α for which the equation is *inconsistent*.

(d) Determine all values of α for which the equation has *exactly one solution*.

(e) Find the solution set of the equation for $\alpha = 0$.

REQUIRED KNOWLEDGE: **Gauss-elimination, row operations, notions of consistency/inconsistency.**

SOLUTION:

First, we form the augmented matrix

$$\left[\, A \,\vdots\, b \,\right] = \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 2 & 2 & 0 & -2 & 10 & \vdots & 0 \\ -1 & 0 & -1 & 2 & -5 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix}$$

Then, we perform row operations to put the augmented matrix into the *row echelon form*:

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 2 & 2 & 0 & -2 & 10 & \vdots & 0 \\ -1 & 0 & -1 & 2 & -5 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix} \xrightarrow{\textbf{2nd}=-2\times\textbf{1st}+\textbf{2nd}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ -1 & 0 & -1 & 2 & -5 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ -1 & 0 & -1 & 2 & -5 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix} \xrightarrow{\textbf{3rd}=1\times\textbf{1st}+\textbf{3rd}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 1 & 8 & -7 & 6 & -1 & \vdots & -6 \end{pmatrix} \xrightarrow{\mathbf{4th}=-1\times\mathbf{1st}+\mathbf{4th}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & -2 & 2 & -2 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix} \xrightarrow{\mathbf{2nd}=-\frac{1}{2}\times\mathbf{2nd}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 2 & -2 & 2 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix} \xrightarrow{\mathbf{3rd}=-2\times\mathbf{2nd}+\mathbf{3rd}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \\ 0 & 6 & -6 & 6 & -6 & \vdots & -6 \end{pmatrix} \xrightarrow{\mathbf{4th}=-6\times\mathbf{2nd}+\mathbf{4th}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \\ 0 & 0 & 0 & 0 & -6 & \vdots & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \\ 0 & 0 & 0 & 0 & -6 & \vdots & -6 \end{pmatrix} \xrightarrow{\text{interchange } \mathbf{3rd} \text{ and } \mathbf{4th}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & -6 & \vdots & -6 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & -6 & \vdots & -6 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \end{pmatrix} \xrightarrow{\mathbf{3rd}=-\frac{1}{6}\times\mathbf{3rd}} \begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \alpha \end{pmatrix}.$$

Then, we have two possible case depending on α.

case 1: $\alpha = 0$
For this case, we obtain

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix}. \tag{1}$$

case 2: $\alpha \neq 0$

In this case, we obtain

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 1 \end{pmatrix}. \tag{2}$$

by multiplying the **4th** row by $\frac{1}{\alpha}$.

1a: From (1) or (2), we see that the *lead* variables are x_1, x_2, and x_5; and the *free* are x_3 and x_4.

1b: If $\alpha \neq 0$, the linear equation corresponding to the equivalent system given by (2) is *inconsistent*. If $\alpha = 0$, then the equivalent system (1) is *consistent* and have *infinitely many solutions* as there are free variables (x_3 and x_4).

1c: It follows from (2) that the equation is *inconsistent* if $\alpha \neq 0$. Otherwise, i.e. when $\alpha = 0$, it is consistent.

1d: Since we obtain free variables whenever the equation is consistent, the equation has never *exactly one solution*.

1e: When $\alpha = 0$, we can perform row operations on the equivalent system given by (2) in order to obtain the reduced row echelon form as follows:

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 & \vdots & 0 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \xrightarrow{\textbf{1st}=-5\times\textbf{3rd}+\textbf{1st}} \begin{pmatrix} 1 & 2 & -1 & 0 & 0 & \vdots & -5 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 0 & \vdots & -5 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \xrightarrow{\textbf{1st}=-2\times\textbf{2nd}+\textbf{1st}} \begin{pmatrix} 1 & 0 & 1 & -2 & 0 & \vdots & -5 \\ 0 & 1 & -1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix}.$$

Now, we can solve the last system of equations:

$$\begin{aligned} x_1 &= -5 - x_3 + 2x_4 \\ x_2 &= x_3 - x_4 \\ x_5 &= 1. \end{aligned}$$

Equivalently, the general solution can be given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} p - 2q - 5 \\ p - q \\ p \\ q \\ 1 \end{pmatrix}$$

where p and q are real numbers.

Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}$$

nonsingular? If so, find its inverse.

REQUIRED KNOWLEDGE: **nonsingular matrices, inverse of a matrix.**

SOLUTION:

We can check whether a given matrix is nonsingular in many ways. For example, we can check whether its determinant is zero or not. Since we need to find the inverse of the given matrix if it is nonsingular in this problem, we can opt for row operations:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 4 & 9 & 16 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\textbf{2nd}=-2\times\textbf{1st}+\textbf{2nd}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 1 \\ 4 & 9 & 16 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 1 \\ 4 & 9 & 16 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\textbf{3rd}=-4\times\textbf{1st}+\textbf{3rd}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 5 & 12 & -4 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 5 & 12 & -4 & 0 & 1 \end{array}\right) \xrightarrow{\textbf{3rd}=-5\times\textbf{2nd}+\textbf{3rd}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 6 & -5 & 1 \end{array}\right)$$

At this stage, we can already conclude that the matrix is nonsingular. To obtain its inverse, we continue performing row operations:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 6 & -5 & 1 \end{array}\right) \xrightarrow{\textbf{3rd}=-\frac{1}{2}\times\textbf{2nd}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{\textbf{2nd}=-2\times\textbf{3rd}+\textbf{2nd}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -8 & 6 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -8 & 6 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{\textbf{1st}=-1\times\textbf{3rd}+\textbf{1st}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -8 & 6 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -8 & 6 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{\textbf{1st}=-1\times\textbf{2nd}+\textbf{1st}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -8 & 6 & -1 \\ 0 & 0 & 1 & 3 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$

Therefore, we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & -\frac{7}{2} & \frac{1}{2} \\ -8 & 6 & -1 \\ 3 & -\frac{5}{2} & \frac{1}{2} \end{pmatrix}.$$

3 (20 pts) **Determinants**

Let a, b, and c be real numbers. Show that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (a-b)(b-c)(c-a).$$

Required Knowledge: **determinants, cofactor expansion.**

Solution:

There are various ways of computing the determinant of a matrix. One way is to use the cofactor expansion, for instance with respect to the first row:

$$\begin{aligned} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} b & c \\ b^2 & c^2 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} a & c \\ a^2 & c^2 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} a & b \\ a^2 & b^2 \end{pmatrix} \\ &= (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b). \end{aligned}$$

Note that

$$\begin{aligned} (a-b)(b-c)(c-a) &= abc - a^2b - ac^2 + a^2c - b^2c + ab^2 + bc^2 - abc \\ &= -a^2b - ac^2 + a^2c - b^2c + ab^2 + bc^2 \\ &= (bc^2 - b^2c) - (ac^2 - a^2c) + (ab^2 - a^2b). \end{aligned}$$

Another possibility is to perform row operations:

$$\begin{aligned} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} &= \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \qquad \begin{matrix} [\mathbf{2nd} = -a \times \mathbf{1st} + \mathbf{2nd}] \\ [\mathbf{3rd} = -a^2 \times \mathbf{1st} + \mathbf{3rd}] \end{matrix} \\ &= \det \begin{pmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{pmatrix} \\ &= (b-a)(c-a) \det \begin{pmatrix} 1 & 1 \\ b+a & c+a \end{pmatrix} \\ &= (b-a)(c-a)(c+a-b-a) \\ &= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a). \end{aligned}$$

Consider the vector space P_4.

(a) Show that $\{ p \in P_4 \mid p(0) = 1 \}$ is *not* a subspace of P_4.

(b) Show that $\{ p \in P_4 \mid p(1) = 0 \}$ is a subspace of P_4. Find a basis for this subspace. What is the dimension of this subspace?

REQUIRED KNOWLEDGE: **vector spaces, subspaces, basis, dimension.**

SOLUTION:

4a: Let $p_1 \in P_4$ and $p_2 \in P_4$ be two polynomials with $p_1(0) = p_2(0) = 1$. Then, we have

$$(p_1 + p_2)(0) = p_1(0) + p_2(0) = 2 \neq 1.$$

In other words, the set $\{ p \in P_4 \mid p(0) = 1 \}$ is *not* closed under addition. Therefore, it is *not* a subspace. Alternatively, let $p \in P_4$ with $p(0) = 1$. Then, we have

$$(2p)(0) = 2p(0) = 2 \neq 1.$$

Consequently, the set is *not* closed under scalar multiplication and hence is *not* a subspace.

4b: To begin with observe that the zero polynomial is an element of the set $\{ p \in P_4 \mid p(1) = 0 \}$. Hence, it is non-empty. Now, let $p \in P_4$ with $p(1) = 0$. Note that

$$(\alpha p)(1) = \alpha p(1) = 0.$$

Therefore, the set is closed under scalar multiplication. Now, let p and q be polynomials having degree less than 4 with $p(1) = q(1) = 0$. Note that

$$(p + q)(1) = p(1) + q(1) = 0.$$

Hence, the set is closed under addition. Consequently, the set $\{ p \in P_4 \mid p(1) = 0 \}$ is a subspace of P_4.

To find a basis for this subspace, let $p(x) = ax^3 + bx^2 + cx + d$. Observe that

$$p(1) = 0 \quad \text{if and only if} \quad a + b + c + d = 0.$$

Therefore, this subspace consists of polynomials of the form

$$p(x) = ax^3 + bx^2 + cx - (a + b + c). \tag{3}$$

Now, we claim that the vectors $x^3 - 1, x^2 - 1, x - 1$ form a basis for this subspace. First, note that these vectors belong to the subspace. To prove that they form a basis, we need to show that they are linearly independent and they span the subspace $\{ p \in P_4 \mid p(1) = 0 \}$. Let α, β, and γ be real numbers such that

$$\alpha(x^3 - 1) + \beta(x^2 - 1) + \gamma(x - 1) = 0.$$

This yields

$$\alpha x^3 + \beta x^2 + \gamma x - (\alpha + \beta + \gamma) = 0$$

and hence $\alpha = \beta = \gamma = 0$. Thus, we showed that the vectors $x^3 - 1, x^2 - 1, x - 1$ are linearly independent. To prove the rest, let $p \in P_4$ with $p(1) = 0$. It follows from (3) that p is of the form

$$p(x) = ax^3 + bx^2 + cx - (a + b + c).$$

Note that

$$p(x) = a(x^3 - 1) + b(x^2 - 1) + c(x - 1).$$

Therefore, the vectors $x^3 - 1, x^2 - 1, x - 1$ span the subspace $\{ p \in P_4 \mid p(1) = 0 \}$. Consequently, they form a basis for the subspace $\{ p \in P_4 \mid p(1) = 0 \}$.

Of course, there are infinitely many other choices for the basis vectors. For instance, the vectors $(x - 1)^3, (x - 1)^2, (x - 1)$ form another basis and so do the vectors $x^3 - x^2, x^2 - x, x - 1$.

By definition, dimension of a subspace is the cardinality of its basis vectors. Therefore, the dimension of the subspace $\{ p \in P_4 \mid p(1) = 0 \}$ is equal to 3.